FOUR - POINT FUNCTIONS OF CHIRAL PRIMARY OPERATORS IN $N = 4$ SYM

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We discuss recent progress in the determination of correlators of chiral primary operators in $N = 4$ Super-Yang-Mills theory, based on a combination of superconformal covariance arguments in $N = 2$ harmonic superspace, and Intriligator's insertion formula. Applying this technique to the calculation of the supercurrent four - point function we obtain a compact and explicit result for its three-loop contribution with comparatively little effort.

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As is well-known, unbroken conformal invariance imposes strong constraints on correlators in quantum field theory. More recently, the analysis of these constraints has been extended to the case of superconformal invariance in Super-Yang-Mills theories by P. Howe and P. West [1]. The natural objects to consider in this context are finite correlators of gauge invariant composite operators, such as the $N = 4$ supercurrent. An independent reason for the study of the same type of correlators has been provided by Maldacena's conjecture which relates them, at leading order in the $\frac{1}{N_c}$ expansion and in the strong coupling limit, to tree-level correlators in AdS supergravity [2]. Yet another motivation for their computation comes from the operator product expansion (see, e.g., A. Petkou's contribution to the present proceedings).

A surprising result of those recent investigations has been the discovery that large classes of such correlators exist which are non-renormalized, i.e. do not receive perturbative corrections at all [3]. Here we consider the simplest such correlator which *does* have radiative corrections, and explicitly compute it at the two- and three- loop level.

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Since no off-shell superspace formulation of $N = 4$ SYM theory is known, in our loop calculations we use a reformulation in terms of $N = 2$ harmonic superfields [4]. Those live on the analytic superspace with coordinates $x_A^{\alpha\dot\alpha}, \theta^{+\alpha}, \bar\theta^{+\dot\alpha}, u_i^\pm$. Here $u_i^\pm$ are the harmonic variables which form a matrix of $SU(2)$ and parametrise the sphere $S^2 \sim SU(2)/U(1)$. A harmonic function $F^{(q)}(u^\pm)$ of $U(1)$ charge q is a function of $u_i^\pm$ invariant under the action of the group $SU(2)$ (which rotates the index i of $u_i^\pm$) and homogeneous of degree q under the action of the group $U(1)$ (which rotates the index $\pm$ of $u_i^\pm$).

The two $N = 2$ ingredients of the $N = 4$ SYM theory are the $N = 2$ SYM multiplet and the $N = 2$ matter (hyper)multiplet. The hypermultiplet is described by an analytic superfield of charge 1, $q^+(x_A, \theta^+, \bar\theta^+, u)$. Its equation of motion is

$$D^{++}q^+ = 0 \tag{1}$$

where D^{++} is the harmonic derivative on S^2 (the raising operator of the group $SU(2)$ realised on the $U(1)$ charges, $D^{++}u^+ = 0,\ D^{++}u^- = u^+$). The equation of motion comes from a Dirac-like action

$$S_{\text{HM}} = -\int dud^4x_A d^2\theta^+ d^2\bar\theta^+\ \tilde q^+ D^{++} q^+ \tag{2}$$

where "~" is an appropriate conjugation. By covariantising this action with respect to a Yang-Mills group with analytic parameters one introduces the SYM gauge potential V^{++}, a charge 2 superfield. Its action can be written as

$$S_{\text{N=2 SYM}} = \frac{1}{4g^2}\int d^4x_L d^4\theta \ \text{tr}\ W^2\ . \tag{3}$$

where $W(x_L, \theta^{i\alpha})$ is the field strength tensor. Unlike the analytic gauge potential, this is a (left-handed) chiral superfield which is harmonic-independent. Its expansion in the gauge potential is an infinite series containing arbitrary powers of V^{++}.

When the hypermultiplet matter is taken in the adjoint representation of the gauge group, the sum of the two actions (2) and (3) describes the $N = 4$ SYM theory,

$$S_{\text{N=4 SYM}} = S_{\text{N=2 SYM}} + S_{\text{HM/SYM}}\ . \tag{4}$$

The Feynman rules derived from this action are formally QCD - like, except that there is an infinite set of gauge self-interactions. In our following three-loop calculation those turn out not to contribute, so that we will have, in fact, only QED - like Feynman diagrams.

We consider the following $N = 2$ correlator

$$G = \langle \text{tr}\tilde q_1^2 \text{tr} q_2^2 \text{tr}\tilde q_3^2 \text{tr} q_4^2 \rangle \tag{5}$$

at the lowest Grassmann level, i.e. with $\bar\theta^+_{1,2,3,4}=0$. In $N=4$ SYM this correlator carries the full information on the correlator of four $N=4$ supercurrents [5] [2] . At the free field (= one-loop) level, it has a trivial contribution

$$G^{1-\mathrm{loop}} \;=\; \frac{16(N_c^2-1)}{(2\pi)^8}\frac{(12)(23)(34)(41)}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2}. \tag{6}$$

where $x_{ij}\equiv x_i-x_j$ and $(kl)\equiv u_k^{+i}u^+_{li}$.

At the two-loop ($O(g^2)$) level the connected graphs are, up to permutations, the ones shown in figure 1.

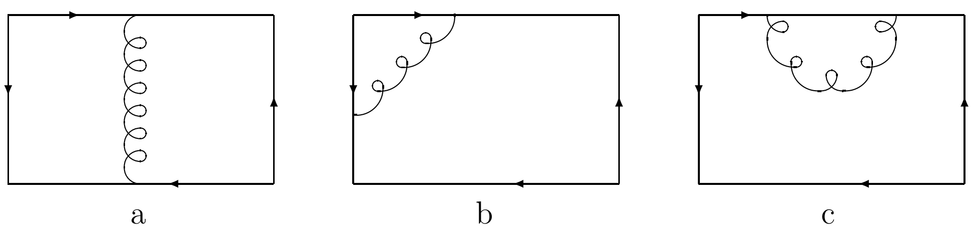

Figure 1: Two-loop graphs

Intriligator's insertion formula [7] allows one to rewrite the sum of graphs as

$$\sum G_i \;=\; -\frac{i}{4}\int dx_5 d^4\theta_5 \sum G_i^{\mathrm{ins}} \tag{7}$$

where G_i^{ins} is a five-point graph obtained from G_i by inserting, into the gauge propagator, a $\frac{\mathrm{tr}W^2}{g^2}$. The five-point function in the integrand is strongly constrained by superconformal covariance, which fixes its complete dependence on the odd variables, and forces it to be of the form $\xi^4F(x,u)$, where ξ^4 is the uniquely determined nilpotent superconformal five-point covariant with the required properties. Defining $\rho_a^{\dot\alpha}\equiv(\theta_5^iu_i^+-\theta_a^+)_\alpha\frac{x_{5a}^{\alpha\dot\alpha}}{x_{5a}^2}$, this covariant can be written as

$$\begin{aligned}\xi^4 \;=\;& (34)^2\rho_1^2\rho_2^2+2(24)(43)\rho_1^2(\rho_2\rho_3)+\frac{4}{3}\Big[(23)(41)+(12)(34)\Big](\rho_1\rho_3)(\rho_2\rho_4)\\ &+\;\text{permutations}\end{aligned} \tag{8}$$

Moreover, after the insertion our graphs factorize into "building blocks" which can be easily computed, and yield expressions which are *rational in the space-time variables* [8].
For example, the building block shown in fig. 2 can be written as

[2] Note added: A rigorous proof of this fact has been given only after the conference [6].

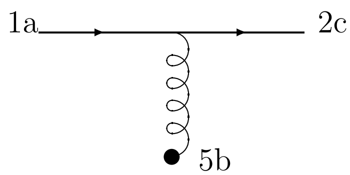

Figure 2: Building block "I"

$$I = \frac{2gf_{abc}}{(2\pi)^4}\frac{(21^-)\rho_1^2 + (12^-)\rho_2^2 - 2(\rho_1\rho_2)}{x_{12}^2} \tag{9}$$

Summing up all graphs one reproduces the known result [9],

$$G^{2-\mathrm{loop}} = -32\frac{ig^2(N_c^2-1)N_c}{(2\pi)^{12}}\frac{R'}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2}h^{(1)}(x_1,x_2,x_3,x_4) \tag{10}$$

where

$$R' = -\frac{1}{2}(12)^2(34)^2\Big[x_{12}^2x_{34}^2 - x_{13}^2x_{24}^2 - x_{14}^2x_{23}^2\Big] + \text{permutations} \tag{11}$$

The remaining integral over the insertion point is the well-known one-loop box integral $h^{(1)} \equiv \int dx_5 \frac{1}{x_{51}^2x_{52}^2x_{53}^2x_{54}^2}$, which can be expressed in terms of logarithms and dilogarithms of the conformal cross ratios.

Proceeding to the three-loop level, a repeated application of the insertion formula allows us to write

$$G^{3-\mathrm{loop}} = -\frac{1}{32}\int d^4x_5d^4\theta_5\int d^4x_6d^4\theta_6\Big\langle \mathrm{tr}\tilde{q}_1^2\mathrm{tr}q_2^2\mathrm{tr}\tilde{q}_3^2\mathrm{tr}q_4^2\mathrm{tr}\frac{W_5^2}{g^2}\mathrm{tr}\frac{W_6^2}{g^2}\Big\rangle \tag{12}$$

Similarly to the two-loop case, the superconformal covariance of the integrand six-point function turns out to require it to be of the form $\xi^4\psi^4F(x,u)$, where ξ^4 is our covariant above, and ψ^4 the corresponding covariant referring to point 6. This factorization is very useful since it means that, instead of computing the component $\sim \theta_5^4\theta_6^4$ of the six-point correlator, which is the one that actually saturates the integrals $\int d^4\theta_5 \int d^4\theta_6$, we can easily infer this component from any other one. As it turns out, the component which is the easiest one to compute in the explicit graph calculation is the "opposite" one defined by

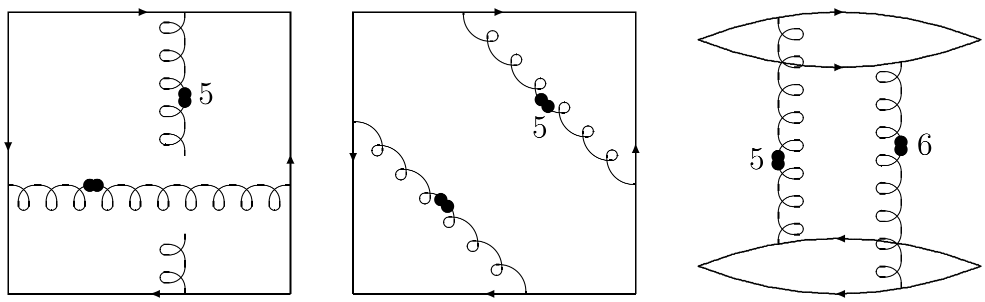

Figure 3: Three-loop graphs

instead setting the chiral Grassmann variables to zero. Of fifteen different graph topologies at the three-loop level only the three graphs shown in fig. 3 contribute to this component.

Since those involve only the building block "I" which is already known from the two-loop calculation, this immediately reduces the original number of four integrations to the two integrals over the insertion points. Moreover, it turns out that, after summing up all terms, only two different integrals remain, namely the one-loop box integral $h^{(1)}$ above and the two-loop integral $h_{12}^{(2)} \equiv x_{12}^2 \int dx_5 \int dx_6 \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{56}^2 x_{16}^2 x_{26}^2 x_{46}^2}$, which is also known explicitly in terms of polylogarithms up to fourth order [10]. Thus only a bit of algebra is required to arrive at the following final result for this three-loop correlator [11]:

$$\begin{aligned} G^{3-\text{loop}} \quad = \quad & -16 \frac{g^4}{(2\pi)^{16}} \frac{(N_c^2-1)N_c^2 R'}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \Big[(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)(h^{(1)})^2 \\ & + 4(h_{12}^{(2)} + h_{13}^{(2)} + h_{14}^{(2)}) \Big] \end{aligned} \tag{13}$$

This result has been confirmed by an $N = 1$ superfield calculation in [12].

To summarize, we have used $N = 2$ harmonic superspace for computing the three-loop correlator of four supercurrents in $N = 4$, reaching a compact and explicit result in terms of polylogarithms of the conformal cross ratios. The computational effort in this calculation was relatively small, due to a fortuitious interplay between superconformal covariance arguments and Intriligator's insertion formula. Considering the fact that the insertion formula by itself seems almost a triviality at the path integral level [8], its usefulness in the present context is quite remarkable: Firstly, the fact that it involves *chiral* insertion points has allowed us to apply superconformal covariance arguments in a way which would not have been possible for the original, purely analytic amplitude, and led to a reduction in the number of Feynman diagrams. Secondly, for the few remaining diagrams we encountered, due to the factorization

into simple building blocks, only twofold integrals instead of the fourfold integrals which had to be computed in the direct approach of [12]. Those twofold integrals were, moreover, *individually* conformal, which explains why only $h^{(1)}$ and $h^{(2)}$ appeared; those are the only finite and conformally covariant integrals which one can build with the number of propagators available at the three-loop level. Those two integrals are (as we indicated already by the superscript) just the first two elements of the infinite series of conformal "multi-ladder" integrals $h^{(k)}$ defined and computed in [10]. This raises the interesting possibility that the n - loop contribution to this correlator may perhaps have the simple form $P_n(g^2 h^{(1)}, g^4 h^{(2)}, \ldots, g^{2n-2} h^{(n-1)})$ with some polynomial P_n.

We believe that even the corresponding four-loop calculation would not be exceedingly difficult with the technique presented here. Knowing the four-loop contribution would be interesting also from the point of view of the Maldacena conjecture, since for this correlator subleading color diagrams appear first at this loop level.

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